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Interpolation Projections and Banach Spaces

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I. INTRODUCTION

Let K be a compact Hausdorff space and $C(K)$ the space of all real or complex valued continuous functions on K. Let $F(K)$ be another Banach space of continuous functions on K, continuously imbedded into $C(K)$:

$$J: F(K) \hookrightarrow C(K).$$

In this paper, we are concerned with the convergence of linear interpolation projections in $F(K)$. We will need the following notation: If A is a topological space with a topology τ and A_n is a sequence of subsets of A, then

$$\tau\text{-Lim}\, A_n := \{a \in A: a = \tau\text{-lim}\, a_n, a_n \in A_n\}.$$

Let $\Delta_n = \{t_j^{(n)}\}_{j=1}^n \subset K$ be a set of distinct points in K. Then the interpolation projection $p(\Delta_n)$ is a linear continuous operator from $F(K)$ into $F(K)$ such that $p^2(\Delta_n) = p(\Delta_n)$ and such that

$$[f - p(\Delta_n)f]\,|_{\Delta_n} = 0, \qquad \forall f \in F(K), \text{dim range } p = n.$$

Every interpolation projection can be written as

$$p(\Delta_n)f = \sum_{j=1}^{n} f(t_j^{(n)})\, s_j^{(n)}, s_j^{(n)} \in F(K); \qquad s_j^{(n)}(t_i^{(n)}) = \delta_{ij}.$$

Thus, the sets $\{s_j(t)\}_{j=1}^n$ and Δ_n characterizes the range and the kernel of $p(\Delta_n)$.

The following is the simple consequence of the uniform boundedness principle:

PROPOSITION 1. *The sequence $p(\Delta_n)$ converges to the identity in $F(K)$ iff*

$$\|p(\Delta_n)\|_{F \to F} \qquad \textit{uniformly bounded,} \tag{1}$$

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and

$$F\text{-Lim range } p(\Delta_n) = F(K). \tag{2}$$

This proposition gives us a necessary and sufficient condition for the convergence of interpolation projections in terms of their ranges. The dependence on Δ_n (or equivalently on $\ker p(\Delta_n)$) does not enter the proposition explicitly.

One can ask if condition (2) can be replaced by

$$K\text{-Lim } \Delta_n = K, \tag{3}$$

and still have Proposition 1 valid. Unfortunately, this is not in general possible. In fact (cf. [1]), if $F(K) = C(K)$, then there exists a sequence of projections satisfying (1) and (3) for which $p(\Delta_n) \nrightarrow I$ (In Section III of this paper, we will show that such projections exist in *any* Banach space.)

On the other hand, we shall show that suitable strengthening condition (1) in Proposition 1 produces the desired result! First, it is easy to see that if $J: F(K) \hookrightarrow C(K)$ is compact, then (1) and (3) imply

$$\|f - p(\Delta_n)f\|_C \to 0, \qquad \forall f \in C(K). \tag{4}$$

If, in addition, F is dense in C and if the norms $\| p(\Delta_n)\|_{C \to C}$ are uniformly bounded, then

$$\|f - p(\Delta_n)f\|_C \to 0, \qquad \forall f \in C(K).$$

The main purpose of this paper is to show that the compactness of J is not only sufficient *but also necessary* for (4). Thus, the convergence properties of interpolation projections characterize the Banach spaces $F(K)$. This in turn leads to the characterization of all compactly imbedded pairs of Banach spaces $X \hookrightarrow Y$.

Throughout this paper, we will assume that there exists at least one sequence of projections $p(\Delta_n)$ with properties (1) and (3). We also assume the existence of $M > 0$ such that for every $t \in K$, there exists $f \in F(K)$ with

$$f(t) = 1, \ \|f\|_F \leqslant M \qquad \text{for some } t \in \Delta_n.$$

The last condition is satisfied if, for example,

$$1 \in F(K).$$

Remark. The motivation of this paper came from studying spline-interpolation projections defined by means of the minimal norm property. In this case, one does not know the range of the projections and therefore the uniform boundedness does not a priori imply the convergence. Theorem 2 shows that one can blindly conclude the convergence if and *only if* the projections are bounded in a stronger norm.

II. The Main Theorem

THEOREM 2. *Let $\Delta_n \subset K$ be such that*

$$K\text{-Lim}\, \Delta_n = K,$$

and there exists a uniformly bounded sequence of projections $p(\Delta_n)$. Then, the following are equivalent:

(a) *The embedding $J\colon K(F) \hookrightarrow C(K)$ is compact.*

(b) *For every uniformly bounded sequence $p(\Delta_n)$,*

$$\|f - p(\Delta_n) f\|_C \to 0, \qquad \forall f \in F(K).$$

Proof. (a) → (b). For each $t \in K$, let $t_n \in \Delta_n$ with $t_n \to t$. Given any function $f \in F(K)$, we obtain

$$f(t) - (p(\Delta_n) f) = f(t) - f(t_n) + p(\Delta_n) f)(t_n) - (p(\Delta_n) f)(t).$$

Clearly, $f(t) - f(t_n) \to 0$ since f is continuous. The set of functions $\{p(\Delta_n) f\}$ is uniformly bounded in $F(K)$ and thus, it is compact in $C(K)$. By the Arzela–Ascoli theorem, the set $\{p(\Delta_n) f\}$ is equicontinuous, and therefore

$$(p(\Delta_n) f)(t) - (p(\Delta_n) f)(t_n) \to 0.$$

Thus, $p(\Delta_n) f$ converges to f pointwise.

Using the compactness argument one more time, we obtain a convergent subsequence

$$p(\Delta_n) f, \qquad n \in N_1 \subset N.$$

The lim $p(\Delta_n) f$ must coincide with the pointwise limit of $p(\Delta_n) f$ which is the function f itself. Thus, the sequence $p(\Delta_n) f$ has a unique cluster point, and hence, we have the desired conclusion

$$\|f - p(\Delta_n) f\|_C \to 0, \qquad \forall f \in F(K).$$

(b) → (a). Without loss of generality, we assume that $1 \in F(K)$. Given a uniformly bounded sequence of projections,

$$p(\Delta_n) f := \sum_{j=1}^{n} f(t_j^{(n)})\, s_j^{(n)},$$

we can assume that $1 \in \text{range } p(\Delta_n)$, $\forall n$. Indeed if this were not true, we could introduce a new projection

$$p_1(\Delta_n)f = f(t_1)\,\sigma_n + \sum_{j=2}^{n} f(t_j^{(n)})\, s_j^{(n)}, \tag{5}$$

where $\sigma_n = 1 - \sum_{j=2}^{n} s_j^{(n)}$. Since $\sigma_n(t_j^{(n)}) = \delta_{1,i}$, (5) defines an interpolation projection with $1 \in \text{range } p_1(\Delta_n)$. The projections

$$\begin{aligned} p_1(\Delta_n)f &= p(\Delta_n)f - f(t_1^{(n)})\, s_1^{(n)} + f(t_1^{(n)})\,\sigma_n \\ &= p(\Delta_n)f + f(t_1^{(n)})(\sigma_n - s_1^{(n)}) \\ &= p(\Delta_n)f + f(t_1^{(n)})[1 - p(\Delta_n)1] \end{aligned}$$

and

$$\| p_1(\Delta_n)f\|_F \leqslant \| p(\Delta_n)f\|_F + \|f\|_C[\|1\|_F + \|p(\Delta_n)1\|_F].$$

Thus, the uniform boundedness of $\| p(\Delta_n)\|$ implies the uniform boundedness of $\| p_1(\Delta_n)\|$.

Now let $\varepsilon_n > 0$; $\varepsilon_n \to 0$. Consider the functions $z_n \in F(K)$ such that

$$\sup_{f \in B/F} \|f - p(\Delta_n)f\|_C = \|z_n - p(\Delta_n)\, z_n\|_C + \varepsilon_n.$$

Then (since $1 \in \text{range } p(\Delta_n)$) we can find $\alpha_n, \beta \neq 0$ such that the sequence of functions

$$g_n = \alpha_n 1 + \beta z_n$$

is uniformly bounded in the F-norm; $g_n(t_k^{(n)}) = 1$ for some $t_k^{(n)} \in \Delta_n$ and $\beta^{-1} \| g_n - p(\Delta_n)\, g_n\|_C = \|z_n - p(\Delta_n) z_n\|_C$.

Now, we introduce the functions

$$\phi_n = g_n - \sum_{j \neq k} g_n(t_j^{(n)})\, s_j^{(n)}$$

and the projections

$$\hat{p}(\Delta_n)f := \sum_{j \neq k} f(t_j^{(n)})\, s_j^{(n)} + f(t_k^{(n)})\,\phi_n.$$

Since $\phi_n(t_j^{(n)}) = \delta_{jk}$, the operators $p(\Delta_n)$ define interpolation projections. Similar to what was done with the functions σ_n, we consider

$$\begin{aligned} \hat{p}(\Delta_n)f &= p(\Delta_n)f + f(t_k^{(n)})[\phi_k - s_k^{(n)}] \\ &= p(\Delta_n)f + f(t_k^{(n)})[\, g_n - p(\Delta_n)\, g_n].\end{aligned}$$

The uniform boundedness of $\|p(\Delta_n)\|_F$ and $\|g_n\|_F$ implies the uniform boundedness of the norms $\|\hat{p}(\Delta_n)\|$. Hence, by the assumption of Theorem 2,

$$\|p(\Delta_n)f - f\|_C \to 0, \qquad \forall f \in F,$$
$$\|\hat{p}(\Delta_n)f - f\|_C \to 0, \qquad \forall f \in F;$$

and consequently

$$\|p(\Delta_n)f - \hat{p}(\Delta_n)f\|_C \to 0, \qquad \forall f \in F.$$

On the other hand,

$$p(\Delta_n)1 - \hat{p}(\Delta_n)1 = s_k^{(n)} - \phi_n = \sum g_n(t_j^{(n)})\, s_j^{(n)} - g_n = p(\Delta_n)\, g_n - g_n.$$

And so

$$\|p(\Delta_n)\, g_n - g_n\|_C \to 0.$$

By the choice of g_n, this convergence is equivalent to

$$\sup_{f \in B_F} \|p(\Delta_n)f - f\|_C \to 0. \tag{6}$$

Now we can view the operators $p(\Delta_n)$ and the imbedding J as operators from $F(K)$ into $C(K)$. Then (6) means

$$\|p(\Delta_n) - J\|_{F \to C} \to 0.$$

But $p(\Delta_n)$ are finite rank operators and hence the operator J must be compact. Thus (b) → (a). ∎

III. Arbitrary Banach Spaces

Let X be an arbitrary Banach space, and let $K = B_{X^*}$ be a unit ball of the dual space equipped with the weak star topology. Then the Banach space X could be considered as a subspace of $C(K)$. The natural imbedding

$$J: X \hookrightarrow C(K)$$

is certainly continuous but never compact.

Every finite rank projection in X has a form

$$p_n x = \sum_{j=1}^{n} \langle l_j^{(n)}, x \rangle\, s_j^{(n)},$$

where $s_j^{(n)} \in X$; $l_j^{(n)} \in B_{X^*}$. So, p_n can be interpreted as $p(\Delta_n)$ with $\Delta_n = \{l_j^{(n)}\}_{j=1}^{n} \subset K$. An immediate corollary of Theorem 2 is:

PROPOSITION 3. *For a given sequence* $\{l_j^{(n)}\}_{j=1}^n \subset B_{X^*}$ *such that* $w^* - \operatorname{Lim}\{l_j^{(n)}\}_{j=1}^n \supset B_{X^*}$, *either there is no uniformly bounded sequence of projections* $p(\Delta_n)$, *or there exists a uniformly bounded sequence of projections* $p(\Delta_n)$ *which does not converge to the identity operator pointwise.*

In particular, if X is a Hilbert space then the second alternative takes place.

Let Y be another Banach space continuously imbedded into X.

THEOREM 4. *The imbedding* $Y \hookrightarrow X$ *is compact iff every* Y*-uniformly bounded sequence of projections of the form*

$$p_n y = \sum_k \langle l_j^{(n)}, y \rangle s_j^{(n)} \text{ with } l_j^{(n)} \in B_{X^*} \text{ and}$$

$$X - w^* - \operatorname{Lim}\{l_j^{(n)}\}_{j=1}^n = B_{X^*},$$

converges to the identity in the X *norm.*

Proof. It is sufficient to observe that since

$$Y \hookrightarrow X \hookrightarrow C(B_{X^*}),$$

then $Y \hookrightarrow C(B_{X^*})$ and the result follows from Theorem 2. ∎

REFERENCE

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